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WEST BENGAL STATE UNIVERSITY

B.Sc. Honours 4th Semester Examination, 2023

MTMACOR10T-MATHEMATICS (CC10)

RING THEORY AND LINEAR ALGEBRA-I

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any *five* questions from the following: 2×5 = 10

(a) If in a ring R, $a^2 = a$ for all $a \in R$, prove that $a+b=0 \Rightarrow a=b$ for all $a, b \in R$.

(b) Let R be a ring with 1. Show that if R is a division ring, then R has no non-trivial ideal.

(c) Show that the characteristic of an integral domain D is either zero or a prime.

(d) Let f be a homomorphism of a ring R into a ring R'. Prove that $f(R) = \{f(a) : a \in R\}$ is a subring of R'.

(e) Let $S = \{(x, y) : x, y \in \mathbb{R}\}$. For $(x, y) \in S, (s, t) \in S$ and $c \in \mathbb{R}$, define $(x, y)+(s, t) = (x+s, y-t)$ and $c(x, y) = (cx, cy)$. Is S a vector space over $\mathbb{R}$? — Justify.

(f) Let V be a vector space of real matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$W = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V : a+b=0\right\}$. Prove that W is a subspace of V.

(g) Find the dimension of the subspace S of the vector space $\mathbb{R}^3$ given by $S = \{(x, y, z) \in \mathbb{R}^3 : 2x+y-z=0\}$.

(h) Define $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ by $T(f(x)) = f'(x)$, where $f'(x)$ denotes the derivative of $f(x)$. Show that T is a linear transformation.

2. (a) Find all subrings of the ring $\mathbb{Z}$ of integers. 4

(b) Let R be a commutative ring with 1 and M be an ideal of R. Show that M is a maximal ideal if and only if R/M is a field. 4

3. (a) Show that $\mathbb{Z}[\sqrt{3}] = \{a+b\sqrt{3} : a, b \in \mathbb{Z}\}$ is an integral domain but not a field. 2+2

(b) Let $n \in \mathbb{Z}$ be a fixed positive integer. If $\mathbb{Z}/\langle n \rangle$ is a field, then show that n is prime, where $\langle n \rangle = \{qn : q \in \mathbb{Z}\}$ and $\mathbb{Z}/\langle n \rangle = \{a+\langle n \rangle : a \in \mathbb{Z}\}$. 4

4. (a) Prove that the cancellation law holds in a ring $(R,+,\cdot)$ if and only if $(R,+,\cdot)$ contains no divisor of zero. 4

(b) If $(R,+,\cdot)$ is an integral domain of prime characteristic p then prove that $(a+b)^p = a^p + b^p$, for all $a, b \in R$. 4

5. (a) Let A be an ideal of a ring R. Define $f: R \to R/A$ by $f(r) = r + A$, for all $r \in R$. Prove that f is a ring homomorphism. 3

(b) If f is a homomorphism of a ring R into a ring S then prove that $R/\ker f \simeq f(R)$. 5

6. (a) Let W_1, W_2 be two subspaces of a vector space V over a field $\mathbb{F}$. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. 4

(b) Let $W = \{(x, y, z) \in \mathbb{R}^3 : x - 4y + 3z = 0\}$. Show that W is a subspace of $\mathbb{R}^3$. Also find a basis of W. 2+2

7. (a) Let V be a vector space over a field $\mathbb{F}$, with a basis consisting of n elements. Then show that any $n+1$ elements of V are linearly dependent. 4

(b) Let V be a vector space of dimension m and W be a vector space of dimension n over a field $\mathbb{F}$. 4

Prove that $\dim(V/W) = m - n$.

8. (a) Let V and W be the vector spaces over the field F and let $T: V \to W$ be a linear transformation. If V is of finite dimension then prove that 5

$$\dim(V) = \dim(\ker T) + \dim(\operatorname{Im} T)$$

(b) Find the linear transformation $T: R^2 \to R^2$ such that $T(2, 3) = (2, 3)$ and $T(1, 0) = (0, 0)$. 3

9. (a) Let $g(x) = 3 + x$. Let $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U: P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by 4

$$T(f(x)) = f'(x)g(x) + 2f(x) \text{ and } U(a + bx + cx^2) = (a+b, c, a-b).$$

Let β and γ be the standard ordered bases for $P_2(\mathbb{R})$ and $\mathbb{R}^3$ respectively. Compute $[U]_\beta^\gamma$, $[T]_\beta$ and $[UT]_\beta^\gamma$.

(b) Determine whether the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by 4

$$T(a_1, a_2) = (3a_1 - a_2,\ a_2,\ 4a_1)$$

is invertible and justify your answer.

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